Indian Statistical Institute
Semestral examination
Analysis of several variables - MMath I

Max. Marks : 60 Time : 3 hours

State clearly the results that you use. Justify your answers.

(1) Decide whether the following statements are True or False. Give **complete** justications for your answer.
  (a) If all the partial derivatives of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ exist at a point x, then the function f is differentiable at x.
  (b) There exists a continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $f(0) = 1$ and $\text{supp}(f) \subseteq D$ where D is the closed unit disc.
  (c) The oriented 1-simplex $\gamma : Q^1 \longrightarrow \mathbb{R}^2 - 0$ defined by $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$, $t \in Q^1$ is the boundary of a 2-chain.
  (d) Pull back of an exact form is exact. [5 x 4=20]

(2) (a) Suppose $f : U \longrightarrow \mathbb{R}^m$ is a function defined on the open set $U \subseteq \mathbb{R}^n$. Show that f is of class C^1 if and only if all partial derivatives $D_j f_i$ exist and are continuous. [10]
  (b) Put $f(0,0) = 0$ and
$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$
  if $(x,y) \neq (0,0)$. Prove that
    (i) $f, D_1 f, D_2 f$ are continuous on $\mathbb{R}^2$,
    (ii) $D_{12} f$ and $D_{21} f$ exist at every point of $\mathbb{R}^2$ and are continuous except at $(0,0)$. [5+5=10]

(3) (a) Define the notion of the pull back of a form. Let ω be the 2-form on $\mathbb{R}^3$ defined by
$$\omega = x\ dx \wedge dz + (y+1)\ dz \wedge dx + dx \wedge dy.$$
  Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by
$$T(u,v) = (u^2 - v^2, u - v, uv).$$
  Describe the pull back ω_T. [2+2]
  (b) Show that the 1-form
$$\omega = \frac{x\ dy - y\ dx}{x^2 + y^2}$$
  defined on $\mathbb{R}^2 - 0$ is not exact. Is there an open subset U of $\mathbb{R}^2 - 0$ on which ω is exact? Justify. [6+2]
  (c) State Green's theorem. Use Green's theorem to evaluate the integral
$$\int_\gamma (1 + xy^2)\, dx - x^2 y\, dy$$
  where γ is the part of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$. [2+6]